The 7 Most Important Equations for Retirement Income Planning

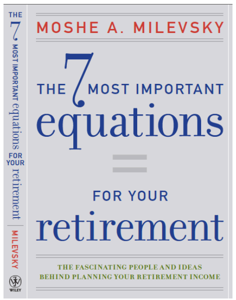
Moshe A. Milevsky

Schulich School of Business
York University
Toronto, CANADA

20 October 2015

My Plan for Next 60 Minutes

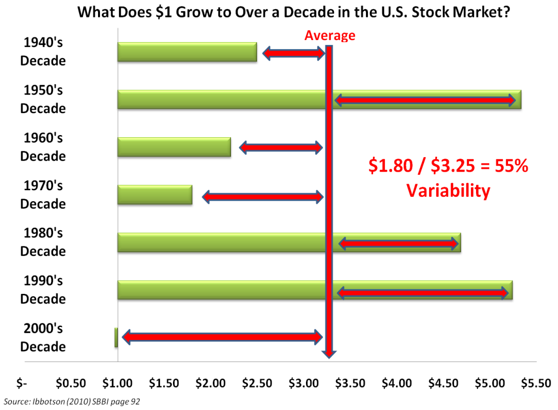
- Discuss mathematical ideas that are needed for proper retirement income planning.
- Focus on the "heroes" associated with these ideas or breakthroughs.
- Touch upon some of my own research work.
- Goal: Help popularize.



QUESTION:
What Number is More Difficult to Forecast?

1. Total **stock market** return during next decade.

2. Total years you will **live/spend** in retirement.



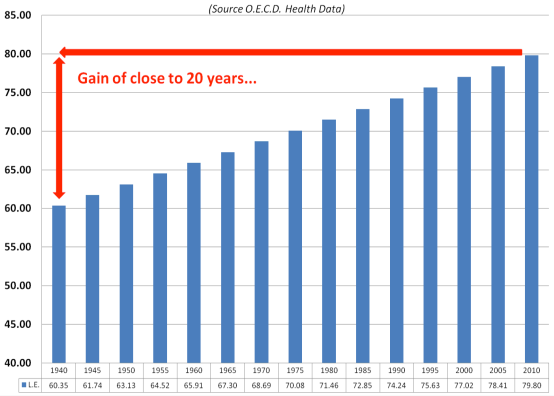
What Does $1 Grow to Over a Decade in the U.S. Stock Market?

Source: Ibbotson (2010) SBBI page 92

The Longevity VIX:
How Random is the Remaining Lifetime of a 65 year-old Retiree?

U.S. Life Expectancy at Birth: Improvement from 1940 to 2010

(Source O.E.C.D. Health Data)



	1940	1945	1950	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005	2010
■ L.E.	60.35	61.74	63.13	64.52	65.91	67.30	68.69	70.08	71.46	72.85	74.24	75.63	77.02	78.41	79.80

These numbers and studies
do not provide any sense of **risk…**

New York Times:
Obituaries…

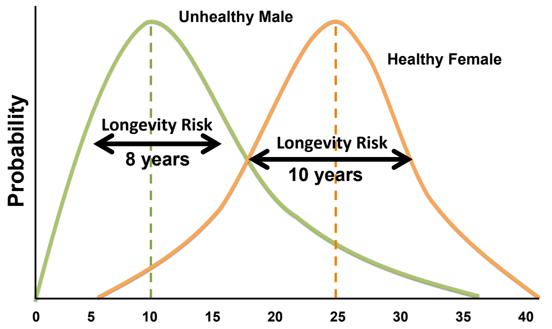
Remaining Life for 65 year-old:

- Arithmetic Mean: = **19.7** years
- Standard Deviation: = **11.0** years
- Variability of: (11)/(19.7) = **56%**

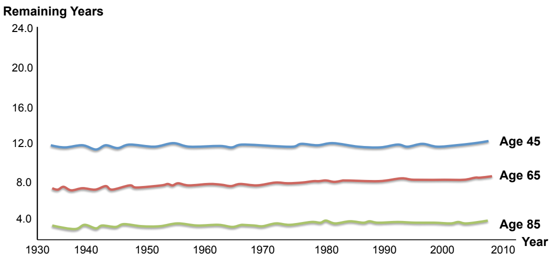
Read the Obituaries: How Many Years in Retirement?				
Mon.	**Tue.**	**Wed.**	**Thu.**	**Fri.**
20.8	2.3	12.1	9.0	34.2
4.3	20.1	30.3	27.5	23.4
20.7	4.7	11.4	20.4	35.6
4.9	20.3	30.2	30.2	33.9
34.5	29.1	19.6	20.9	28.8
18.8	24.3	18.7	19.9	28.3
21.0	30.2	40.5	34.2	17.0
10.3	6.2	4.2	24.0	17.5
8.5	36.3	24.7	11.8	0.3

Your Money or Your Life?
Equally Risky

Remaining Years at Retirement

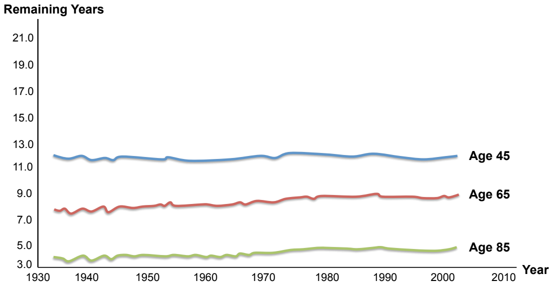


Standard Deviation of Remaining Life in U.S. (Male)



Source: Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org (Data downloaded in Feb. 2012, Calculations by Minjie Zhang, Feb. 2012)

Standard Deviation of Remaining Life in U.S. (Female)



Source: Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org (Data downloaded in Feb. 2012, Calculations by Minjie Zhang, Feb. 2012)

Takeaway: Don't confuse two moments

Mortality Modeling: Who Gets the Credit?

- British Demographer and Actuary
- Fellow of the Royal Society
- Never attended university!
- Brother-in-law of M. Montefiore

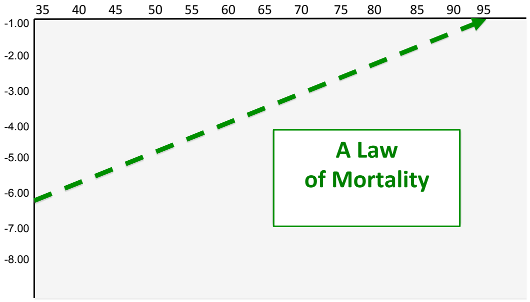
Benjamin Gompertz
(1779-1865)

Longevity in the U.S.

Number of Americans…	
> Age 90	**1,900,000**
> Age 100	**53,600**

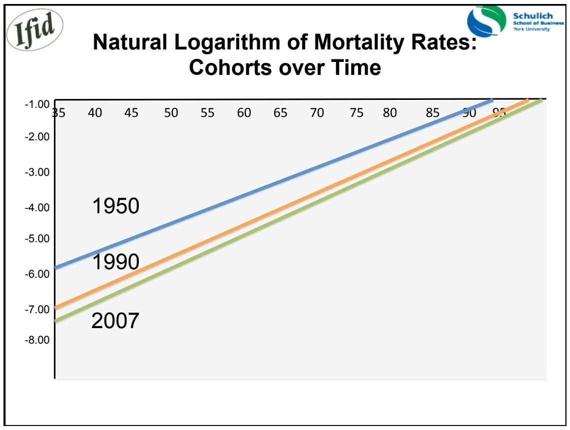
Source: *Census Bureau 2010 (2012)*

Logarithm of Mortality Rates

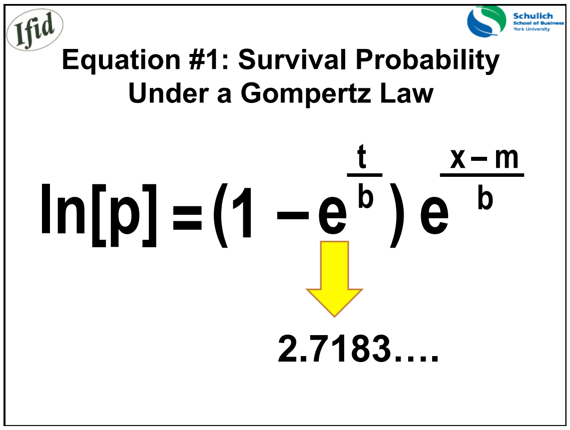


The Gompertz Law of Mortality

The death rate increases every single year of your life by approximately **9%**. Like clockwork, the grim reaper takes **9% more** of your cohort this year, compared to the previous year.



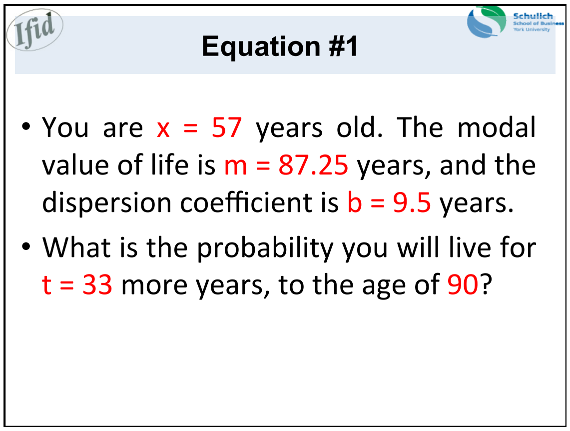
Natural Logarithm of Mortality Rates: Cohorts over Time



Equation #1: Survival Probability Under a Gompertz Law

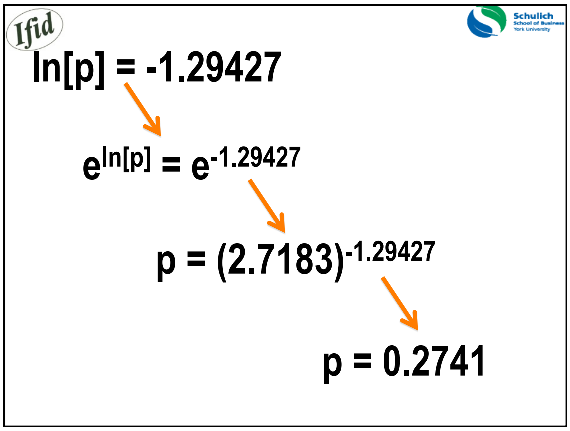
$$\ln[p] = (1 - e^{\frac{t}{b}})\, e^{\frac{x-m}{b}}$$

2.7183….



Equation #1

- You are $x = 57$ years old. The modal value of life is $m = 87.25$ years, and the dispersion coefficient is $b = 9.5$ years.
- What is the probability you will live for $t = 33$ more years, to the age of 90?

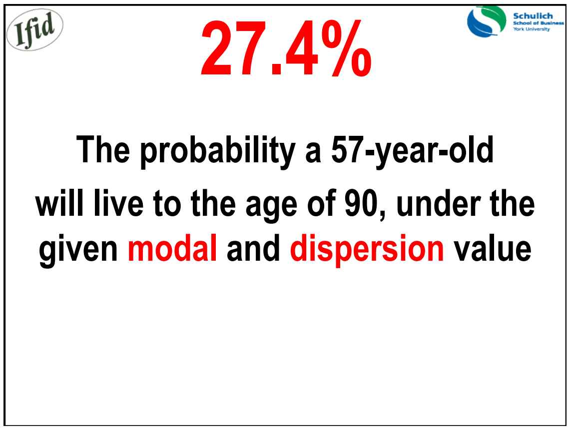


$$\ln[p] = -1.29427$$

$$e^{\ln[p]} = e^{-1.29427}$$

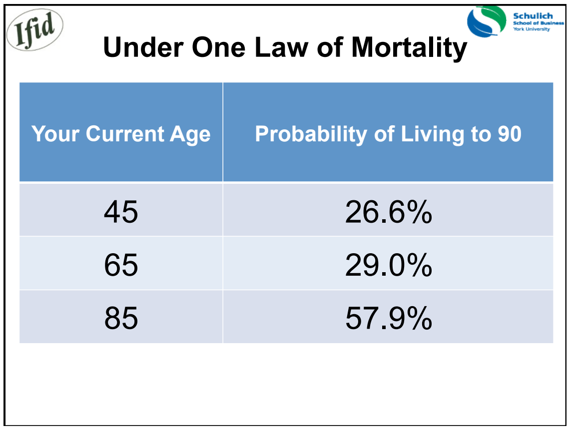
$$p = (2.7183)^{-1.29427}$$

$$p = 0.2741$$



27.4%

The probability a 57-year-old will live to the age of 90, under the given modal and dispersion value



Under One Law of Mortality

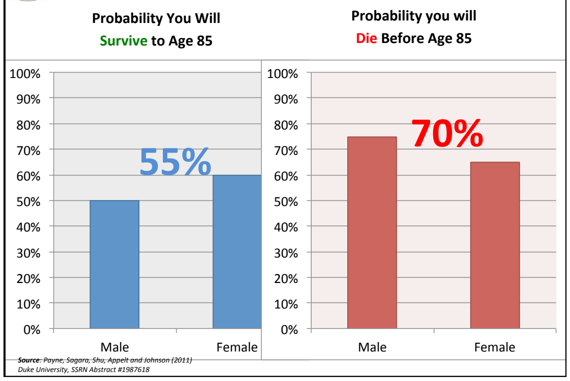
Your Current Age	Probability of Living to 90
45	26.6%
65	29.0%
85	57.9%

Planning and Budgeting for an uncertain horizon isn't easy….

…People underestimate mortality risks from likely causes of death and overestimate mortality risks from unlikely causes of death…

Journal of Experimental Psychology, Human Learning and Memory
1978, Vol. 4(6), pg. 551-578

Does this make sense?



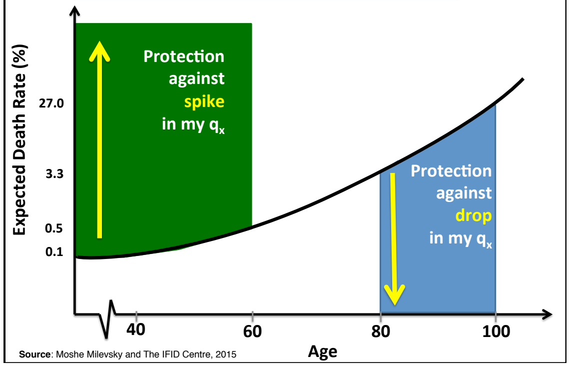
Source: Payne, Sagara, Shu, Appelt and Johnson (2011)
Duke University, SSRN Abstract #1987618

"…Overall, the estimated mean life expectancies, across three studies, were between 7.3 to 9.2 years longer when solicited in ***live-to*** *vs.* ***die-by*** *frame…."*

Source: Payne, Sagara, Shu, Appelt & Johnson, 2011
Life Expectancy as a Constructed Belief

There is an important economic role for insurance over the entire lifecycle.

This curve will kill you….



Source: Moshe Milevsky and The IFID Centre, 2015

Equation #2: Price Your Legacy

$$A_x = \sum_{i=0}^{\infty} \frac{({}_i p_x)(q_{x+i})}{(1+R)^{(i+1)}}$$

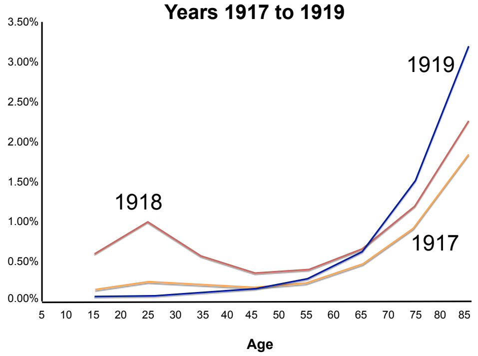
If you *really* want to maximize your legacy, then get some (more) life insurance…

Promoting the Value of Life Insurance: Who Gets the Credit?

Solomon S. Huebner (1882-1964)

- Professor of Insurance, Wharton.
- Founder of the American College
- Promoted the concept of human life value (HLV)
- Consultant to the US Government
- Traveled the world giving lectures on the importance of life insurance.

Mortality Rates During Period of Spanish Flu: Years 1917 to 1919



Now that you appreciate the randomness of human longevity….

What is the longevity of your investment portfolio?

PV of $1 for N periods, discounted at R.

$$PV = \sum_{i=1}^{N} \frac{1}{(1+R)^i} = \frac{1-(1+R)^{-N}}{R}$$

Solve for N – how long will the money last?

Equation #3: Continuous Time

$$L = \frac{1}{g} \ln\left[\frac{w/M}{w/M - g}\right]$$

Using the equation

- You have a $300,000 nest egg, growing at 3% and you want to withdraw $30,000 per year.
- What is the (expected) longevity of your portfolio?

Plug in the numbers…

$$L=\frac{1}{3\%}\ln\left[\frac{\$30/\$300}{\$30/\$300 - 3\%}\right]$$

$$L=\frac{1}{g}\ln\left[\frac{\$w/\$M}{\$w/\$M - g}\right]$$

Plug in the numbers…

$$L=\frac{1}{3\%}\ln\left[1.42857\right]$$

The number we are looking for is …..

L = 11.9 years

But, is a 3% real (inflation adjusted) growth rate realistic?

Longevity of Portfolio in Years

Assuming a real interest rate of **1.5%**

Nest Egg at Retirement "M"

Withdrawal Rate "w"

@ 1.5%	$200,000	$300,000	$400,000
$20,000	10.8	17.0	23.8
$25,000	8.5	13.2	18.3
$30,000	7.0	10.8	14.9
$35,000	6.0	9.2	12.5

Who Deserves the Credit?

Liber Abaci

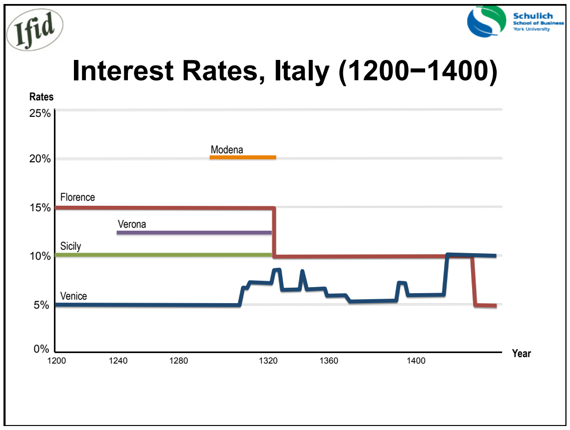
LXVIII
× XVI

1, 2, 3, 5, 8, 13, 21, 34, 55…n

He discovered the technique we still use today for computing present and discounted values.

Fibonacci
(1170-1250)

Leonardo Pisano Fi



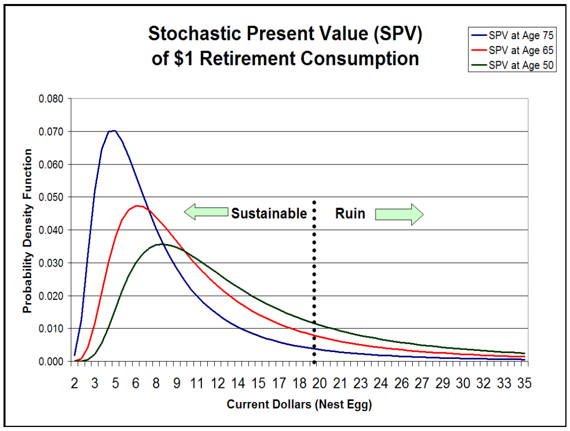
Interest Rates, Italy (1200–1400)

Ok, but what about when the rate of return on investment is random?

Real World Retirement & Dual Randomness

$$PV \approx \frac{1}{(1+\tilde{R}_1)} + \frac{1}{(1+\tilde{R}_1)(1+\tilde{R}_2)} + \ldots + \frac{1}{(1+\tilde{R}_1)(1+\tilde{R}_2)\ldots(1+\tilde{R}_{\tilde{T}})}$$

The length of life ***T*** is random, and the year-over-year investment return ***R*** is random. The present value of your retirement income plan is random



Stochastic Present Value (SPV) of $1 Retirement Consumption

Lifetime Ruin Probability

Spend Retire	$4 per $100	$6 per $100
Age 65	7.6%	22.1%
Age 75	2.5%	9.8%

Parameters: m = 87.25, b = 9.5, μ = 8%, σ = 20%

Equation #4

Lifetime Ruin Probability (LRP)

$P\lambda_t =$

Equation #4

Lifetime Ruin Probability (LRP)

$$P\lambda_t = \frac{\partial P}{\partial t}$$

Equation #4

Lifetime Ruin Probability (LRP)

$$P\lambda_t = \frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} +$$

Equation #4

Lifetime Ruin Probability (LRP)

$$P\lambda_t = \frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} + \frac{1}{2}\sigma^2 w^2$$

Equation #4

Lifetime Ruin Probability (LRP)

$$P\lambda_t = \frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} + \frac{1}{2}\sigma^2 w^2\frac{\partial^2 P}{\partial w^2}$$

Must use numerical techniques to solve this PDE.

Available online at www.sciencedirect.com
SCIENCE @ DIRECT®
Insurance: Mathematics and Economics 34 (2004) 421–447
INSURANCE MATHEMATICS & ECONOMICS
www.elsevier.com/locate/econbase

Ruined moments in your life: how good are the approximations?

H. Huang[a], M.A. Milevsky[b,*], J. Wang[a]

[a] Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ont., Canada M3J 1P3
[b] Schulich School of Business, York University, 4700 Keele Street, Toronto, Ont., Canada M3J 1P3

Received October 2003; received in revised form March 2004; accepted 8 March 2004

Abstract

In this paper we implement numerical PDE solution techniques to compute the *probability of lifetime ruin* which is the probability that a fixed retirement consumption strategy will lead to financial insolvency under stochastic investment returns and lifetime distribution. This problem is a variant of the classical and illustrious ruin problem in insurance, but adapted to individual circumstances.

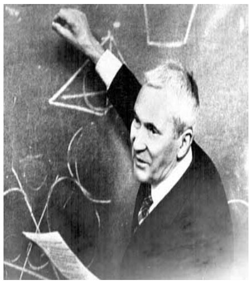
Using equity market parameters derived from US-based financial data we conclude that a 65-year-old retiree requires 30 times their desired annual (real) consumption to generate a 95% probability of sustainability, which is equivalent to a 5% probability of lifetime ruin, if the funds are invested in a well-diversified portfolio. The 30-to-1 margin of safety contrasts with the relevant annuity factor for an inflation-linked pension which would generate a *zero* probability of lifetime ruin.

Our paper then goes on to compare the PDE-based values with moment matching and comonotonic-based approximations that have been proposed in the literature. Our results indicate that the *Reciprocal Gamma* approximation provides an accurate fit as long as the volatility of the underlying investment return does not exceed $\sigma = 30\%$ per annum, which is consistent with capital market history. At higher levels of volatility the moment matching approximations break down. We also confirm that the comonotonic-based lower bound approximation provides remarkably accurate results when the time steps are small though.

Our results should be of interest to academics, practitioners and software developers who are interested in computing sustainable consumption and withdrawal rates towards the end of the human life-cycle, but without resorting to crude simulations.

Keywords: Annuity; Pension; Retirement; Stochastic present value

Retirement Ruin Probability: Who gets the credit?

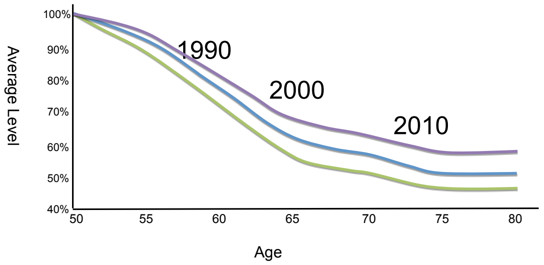


Andrey N. Kolmogorov (1903-1987)

- Russian Mathematician
- Parents were communist revolutionaries.
- It is said: "*What Euclid did for geometry, Kolmogorov did for probability*."
- Awarded Order of Lenin seven (7) times.
- Founded schools for children to study math and sciences.

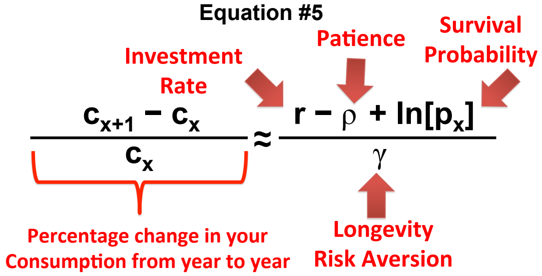
Is it rational to assume a constant spending rate for life?

Retirement Expenditures by Retiring Population: Baseline at Age 50



Economists and financial practitioners have different ways of “thinking” about spending…

Optimal Spending to Maximize Utility Equation #5



Investment Rate

Patience

Survival Probability

$$\underbrace{\frac{c_{x+1} - c_x}{c_x}}_{\text{Percentage change in your Consumption from year to year}} \approx \frac{r - \rho + \ln[p_x]}{\gamma}$$

Longevity Risk Aversion

Financial Analysts Journal
Volume 67 • Number 2
©2011 CFA Institute
FAJ

Spending Retirement on Planet Vulcan: The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates

Moshe A. Milevsky and Huaxiong Huang

Recommendations from the media and financial planners regarding retirement spending rates deviate considerably from utility maximization models. This study argues that wealth managers should advocate dynamic spending in proportion to survival probabilities, adjusted up for exogenous pension income and down for longevity risk aversion.

In our study, we attempted to derive, analyze, and explain the optimal retirement spending policy for a utility-maximizing consumer facing (only) a stochastic lifetime. We deliberately ignored financial market risk by assuming that all investment assets are allocated to risk-free bonds (e.g., Treasury Inflation-Protected Securities [TIPS]). We made this simplifying assumption in order to focus attention on the role of longevity risk aversion in determining optimal consumption and spending rates during a retirement period of stochastic length.

By longevity risk aversion, we mean that different people might have different attitudes toward the “fear” of living longer than anticipated and possibly depleting their financial resources. Some might respond to this economic risk by spending less early on in retirement, whereas others might be willing to take their chances and enjoy a higher

retirement spending behavior has not received as much attention, and most practitioners are unfamiliar with the concept.

Although neither our framework nor our mathematical solution is original—they can be traced back almost 80 years—we believe that the insights from a normative life-cycle model (LCM) are worth emphasizing in the current environment, which has grown jaded by economic models and their prescriptions. Our pedagogical objective was to contrast the optimal (i.e., utility-maximizing) retirement spending policy with popular recommendations offered by the investment media and financial planners.

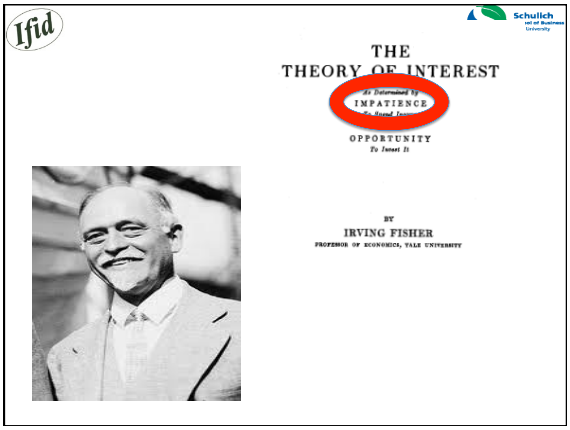
The main results of our investigation are as follows: Counseling retirees to set initial spending from investable wealth at a constant inflation-adjusted rate (e.g., the widely popular 4 percent rule) is consistent with life-cycle consumption

---

Who Gets the Credit?

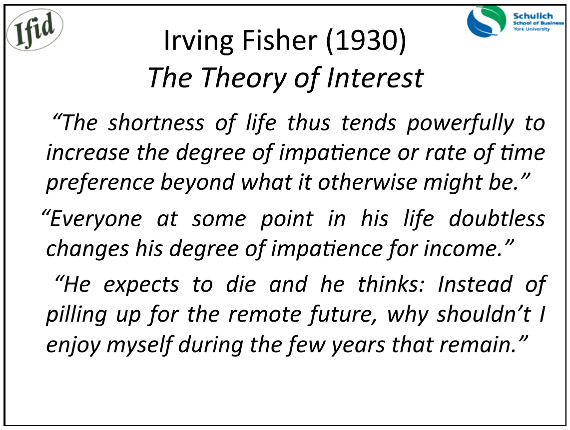
Irving Fisher (1867-1947)

- **Professor of Economics, Yale.**
- **Created first inflation-indices.**
- **Inventor, entrepreneur, spokesperson, health advocate.**
- **Best known for his infamously incorrect forecast of the stock market in 1929.**



THE
THEORY OF INTEREST
As Determined by
IMPATIENCE
To Spend Income
OPPORTUNITY
To Invest It

BY
IRVING FISHER
PROFESSOR OF ECONOMICS, YALE UNIVERSITY

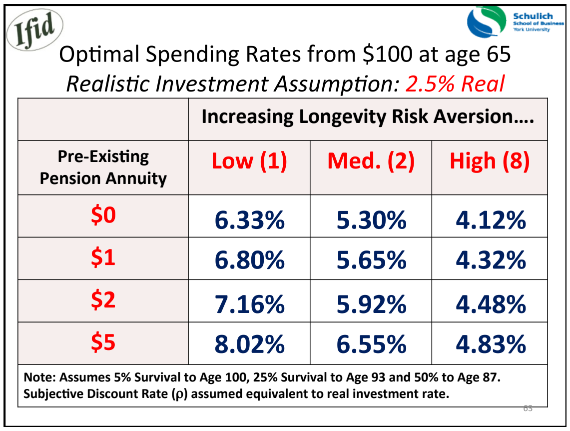


Irving Fisher (1930)
The Theory of Interest

"The shortness of life thus tends powerfully to increase the degree of impatience or rate of time preference beyond what it otherwise might be."

"Everyone at some point in his life doubtless changes his degree of impatience for income."

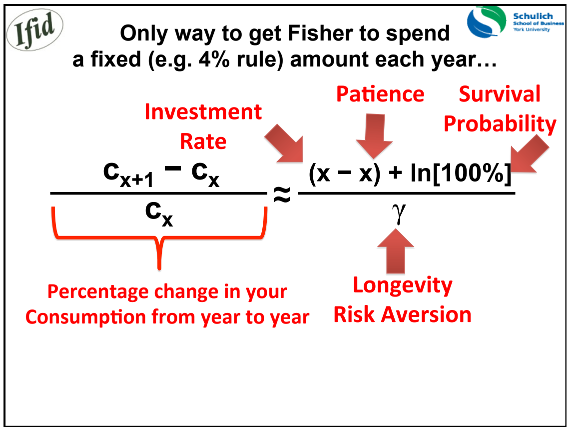
"He expects to die and he thinks: Instead of pilling up for the remote future, why shouldn't I enjoy myself during the few years that remain."



Optimal Spending Rates from $100 at age 65
Realistic Investment Assumption: 2.5% Real

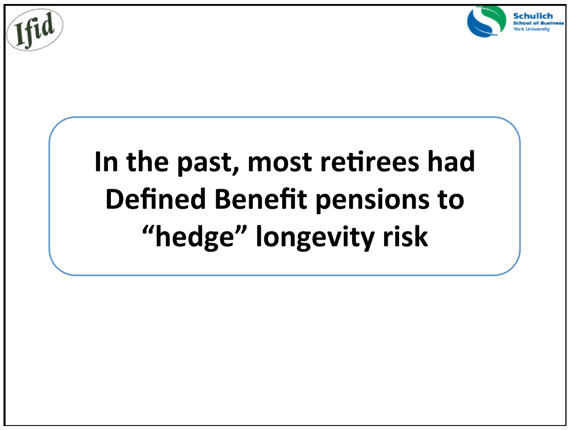
	Increasing Longevity Risk Aversion….		
Pre-Existing Pension Annuity	**Low (1)**	**Med. (2)**	**High (8)**
$0	**6.33%**	**5.30%**	**4.12%**
$1	**6.80%**	**5.65%**	**4.32%**
$2	**7.16%**	**5.92%**	**4.48%**
$5	**8.02%**	**6.55%**	**4.83%**

Note: Assumes 5% Survival to Age 100, 25% Survival to Age 93 and 50% to Age 87. Subjective Discount Rate (ρ) assumed equivalent to real investment rate.

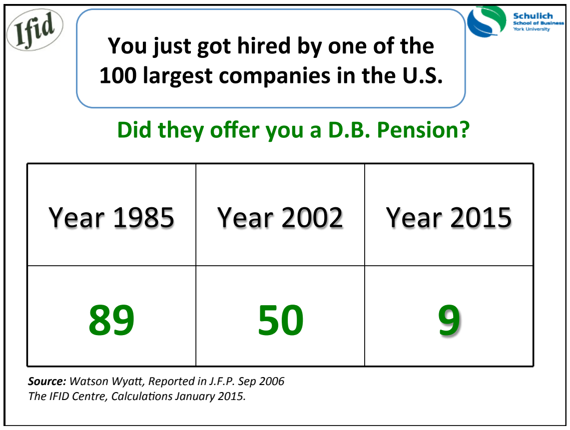


Only way to get Fisher to spend a fixed (e.g. 4% rule) amount each year…

$$\frac{c_{x+1} - c_x}{c_x} \approx \frac{(x - x) + \ln[100\%]}{\gamma}$$



In the past, most retirees had Defined Benefit pensions to "hedge" longevity risk



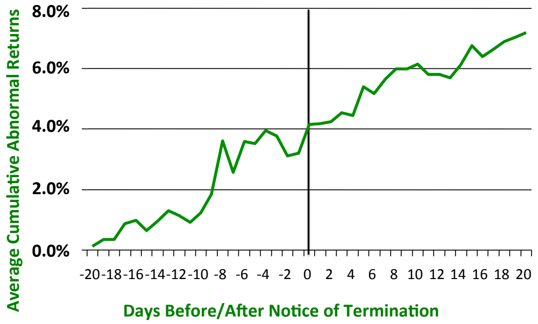
You just got hired by one of the 100 largest companies in the U.S.

Did they offer you a D.B. Pension?

Year 1985	Year 2002	Year 2015
89	**50**	**9**

***Source:** Watson Wyatt, Reported in J.F.P. Sep 2006*
The IFID Centre, Calculations January 2015.

Do Markets Like "Frozen" DBs?



© *The Journal of Risk and Insurance*, 2010, Vol. 77, No. 4, 893-909
DOI: 10.1111/j.1539-6975.2010.01363.x

DO MARKETS LIKE FROZEN DEFINED BENEFIT PENSIONS? AN EVENT STUDY

Moshe A. Milevsky
Keke Song

ABSTRACT

An increasing number of North American companies are freezing or terminating their traditional defined benefit (DB) pension plans. In this article we document a positive announcement effect when a publicly traded company discloses that it has partially or fully frozen its DB plan and replaced it with—or enhanced—the 401(k) defined contribution (DC) plan. This positive risk-adjusted return is greater for firms with higher beta and/or lower return on equity (ROE) prior to the freeze. In other words the positive impact is more pronounced for firms that are likely to face financial distress if they maintain their traditional pension plan and the associated long-term promises.

"At present, the only way a company can manage the risk of long-lived workers is to work them so hard that they die within a few years of retirement; this is not a good way to retain staff."

Financial Times, editorial, September 30, 2006

U.S. TREASURY DEPARTMENT
OFFICE OF PUBLIC AFFAIRS

FOR IMMEDIATE RELEASE: October 24, 2014
CONTACT: Erin Donar, Treasury Public Affairs (202) 622-2690

TREASURY ISSUES GUIDANCE TO ENCOURAGE ANNUITIES IN 401(k) PLANS
New option for plan sponsors to include income annuities in default target date fund investment options

WASHINGTON – In order to help retirees manage their savings and ensure they have a stream of regular income throughout retirement, the U.S. Department of the Treasury and the Internal

The Value or Price of a Life Annuity Equation #6

$$a_x = \sum_{i=1}^{\infty} \frac{{}_t p_x}{(1+r)^i}$$

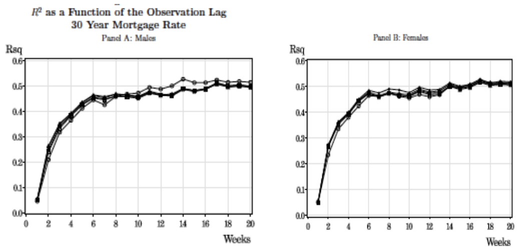
Who Gets the Credit?

Edmond Halley (1656-1742)

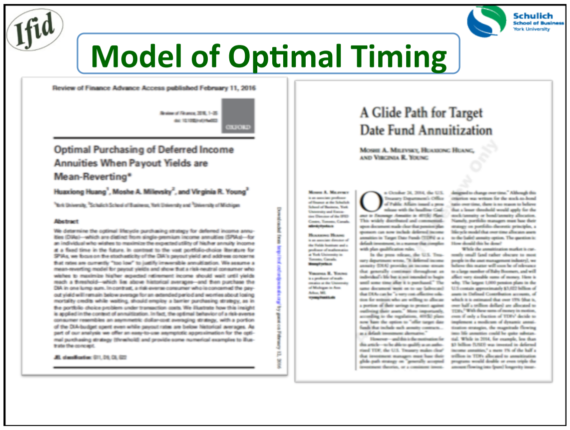
- British Astronomer Royal
- Savilian Professor of Geometry at Oxford University
- Mapped earth's magnetic field
- Isaac Newton's *Principia* publisher

Wrote and published hundreds of papers on astronomy and geophysics, and one paper in 1693 on pricing life annuities!

What Moves Annuity Prices?



By: N. Charupat, M. Kamstra and M.A. Milevsky
SSRN Working Paper #2021579
Journal of Risk and Insurance, September 2016



Model of Optimal Timing

Review of Finance Advance Access published February 11, 2016

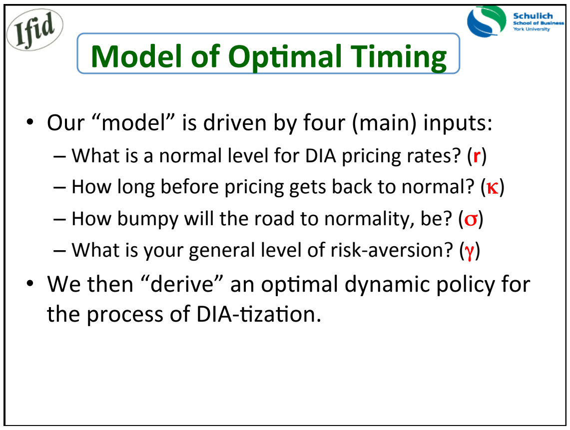
Optimal Purchasing of Deferred Income Annuities When Payout Yields are Mean-Reverting*

Huaxiong Huang[1], Moshe A. Milevsky[2], and Virginia R. Young[3]

Abstract

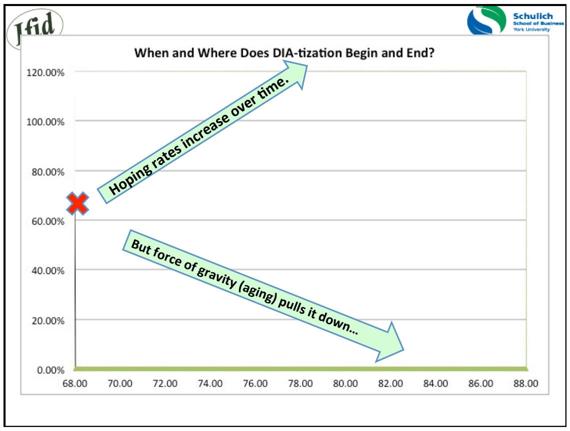
A Glide Path for Target Date Fund Annuitization

Moshe A. Milevsky, Huaxiong Huang, and Virginia R. Young



Model of Optimal Timing

- Our “model” is driven by four (main) inputs:
  - What is a normal level for DIA pricing rates? (r)
  - How long before pricing gets back to normal? (κ)
  - How bumpy will the road to normality, be? (σ)
  - What is your general level of risk-aversion? (γ)
- We then “derive” an optimal dynamic policy for the process of DIA-tization.



When and Where Does DIA-tization Begin and End?

Hoping rates increase over time.

But force of gravity (aging) pulls it down…

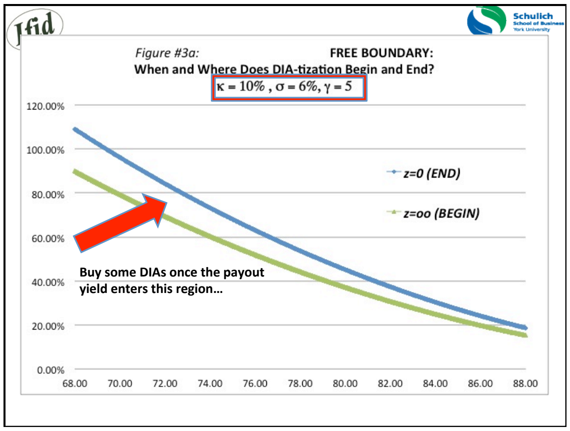
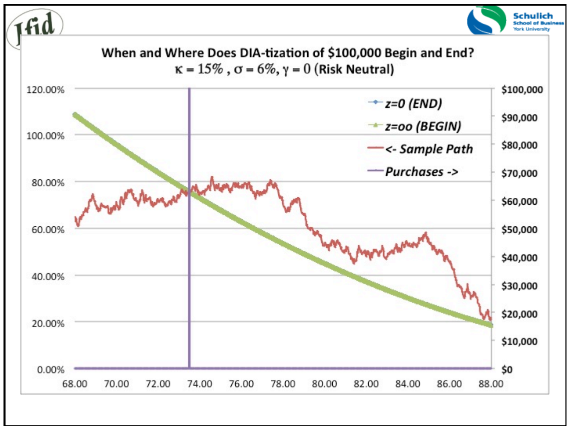
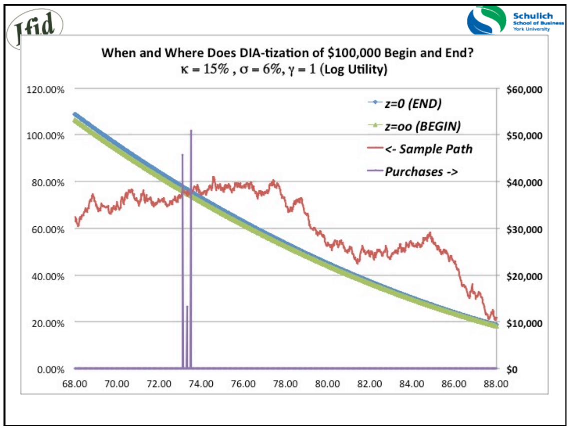


Figure #3a: FREE BOUNDARY:
When and Where Does DIA-tization Begin and End?
$\kappa = 10\%$, $\sigma = 6\%$, $\gamma = 5$

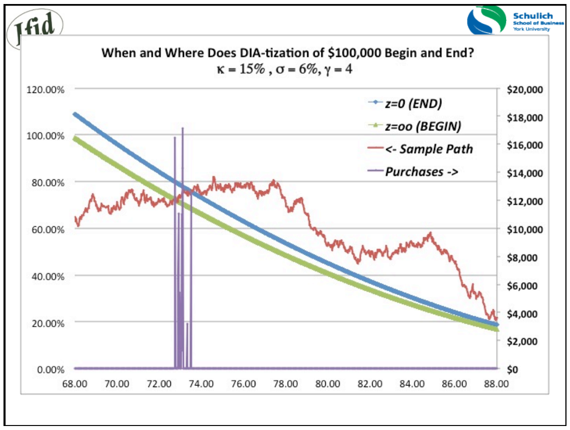
Buy some DIAs once the payout yield enters this region…



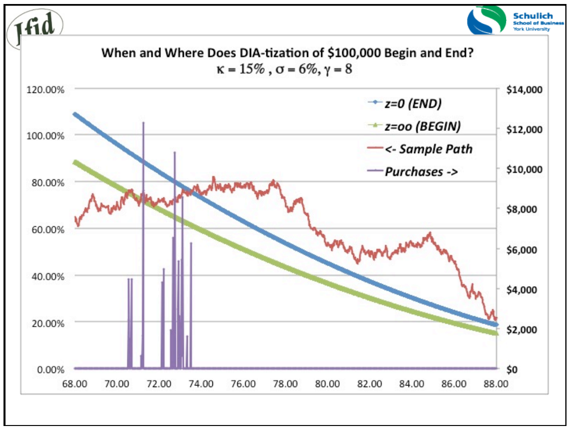
When and Where Does DIA-tization of $100,000 Begin and End?
$\kappa = 15\%$, $\sigma = 6\%$, $\gamma = 0$ (Risk Neutral)



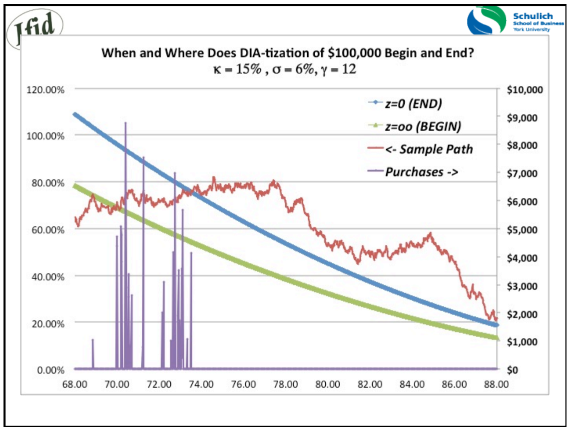
When and Where Does DIA-tization of $100,000 Begin and End?
$\kappa = 15\%$, $\sigma = 6\%$, $\gamma = 1$ (Log Utility)



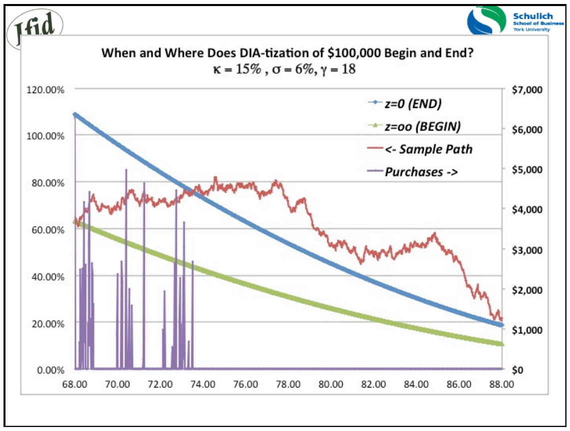
When and Where Does DIA-tization of \$100,000 Begin and End?
$\kappa = 15\%$, $\sigma = 6\%$, $\gamma = 4$



When and Where Does DIA-tization of \$100,000 Begin and End?
$\kappa = 15\%$, $\sigma = 6\%$, $\gamma = 8$



When and Where Does DIA-tization of \$100,000 Begin and End?
$\kappa = 15\%$, $\sigma = 6\%$, $\gamma = 12$

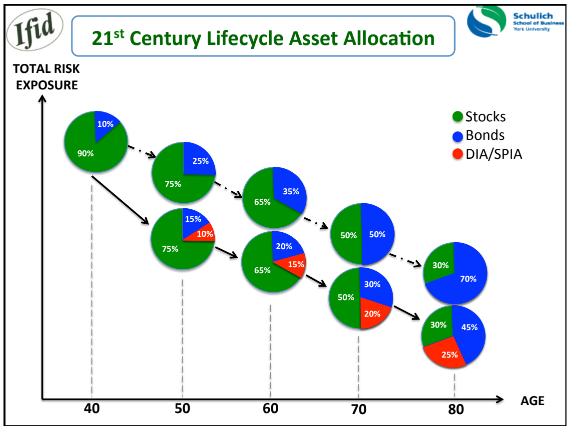


When and Where Does DIA-tization of \$100,000 Begin and End?
$\kappa = 15\%$, $\sigma = 6\%$, $\gamma = 18$

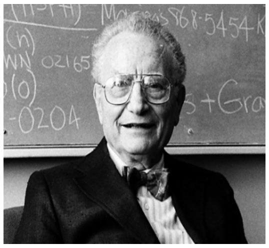
Optimal Timing Takeway

- If you are (truly) **risk-neutral**…wait for rates to get back to normal.
- If you are **moderately risk averse**…then perhaps stick your toe in the water, soon.
- If you are **highly risk averse**…then start a process of dollar cost averaging (DCA) into the deferred income annuity (DIA), immediately.

21st Century Lifecycle Asset Allocation



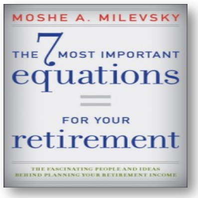
Who Deserves the Final Credit?



Paul Samuelson (1915-2009)

- Professor of Economics MIT
- Nobel Laureate 1970.
- Economic Advisor to J.F.K.
- Many of his students went-on to win Nobel prize.
- Author of most popular textbook in economics.
- Was the first to introduce dynamic programing to asset allocation over the lifecycle.

For the 7th equation… Read the book



Final Words

- There is a core body of mathematical knowledge every *financial advisor* or wealth manager must be aware of – to provide competent and unbiased financial advice.
- This area merges finance, economics, actuarial science, insurance theory and even biology. It is interdisciplinary, but with a firm *mathematical core*.
- The simple question: *Will my money last as long as I do?* can lead to some very deep and interesting mathematical ideas.
- I hope to continue to be part of it (for a long time.)